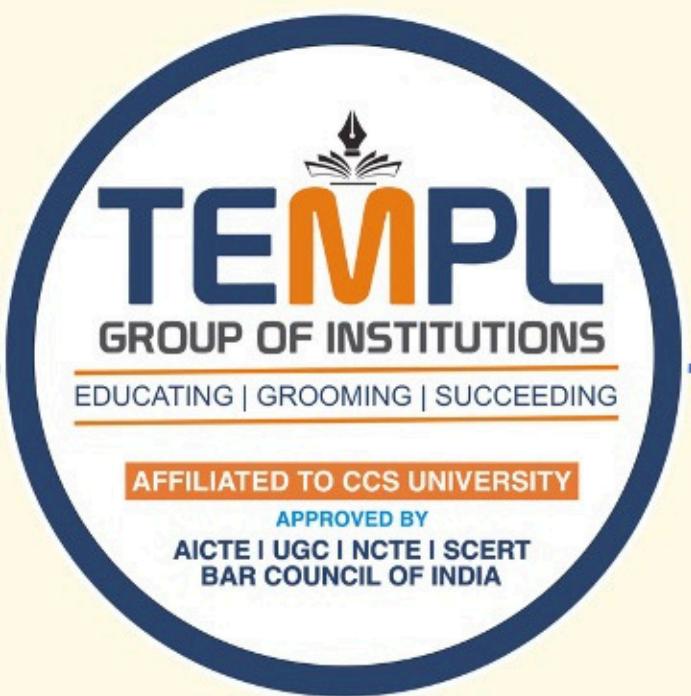


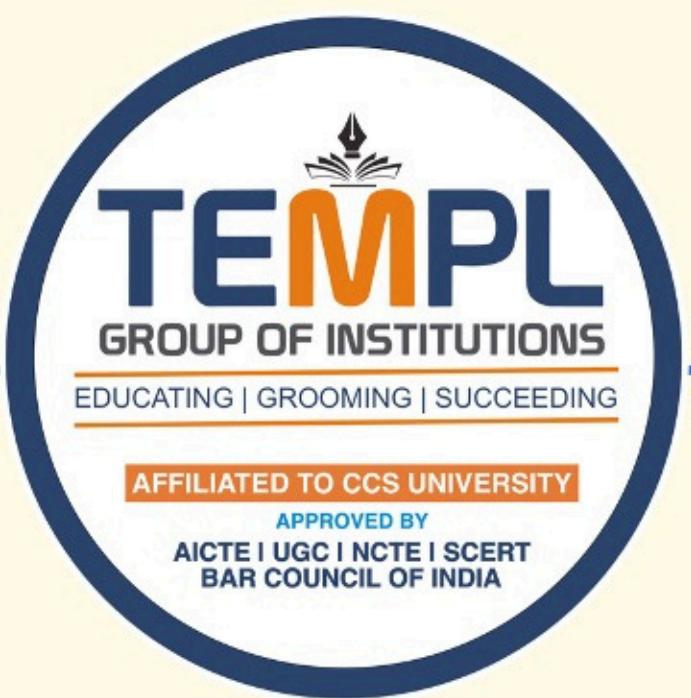
BCA [5TH SEM]
COURSE CODE : BCA-504



NUMERICAL METHODS

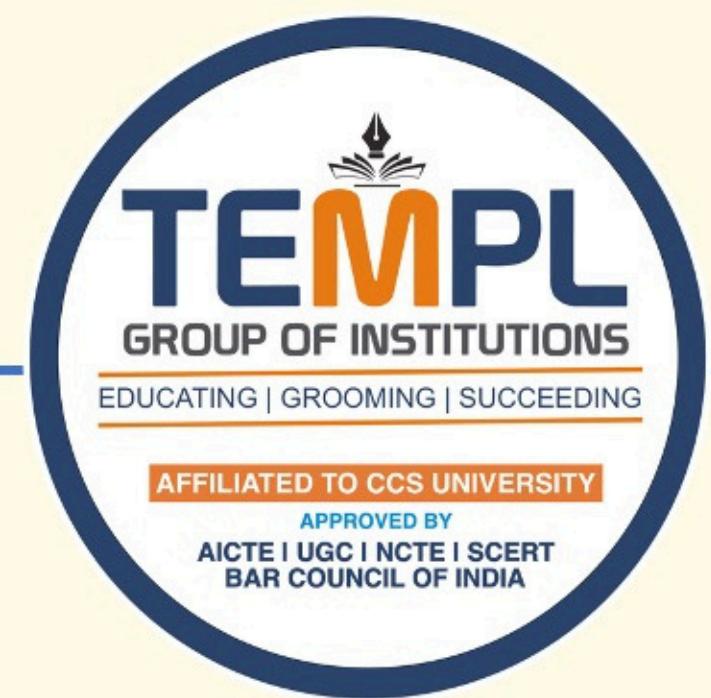
VAISHALI
[Faculty of Mathematics]

UNIT-5



SOLUTION OF DIFFERENTIAL EQUATION

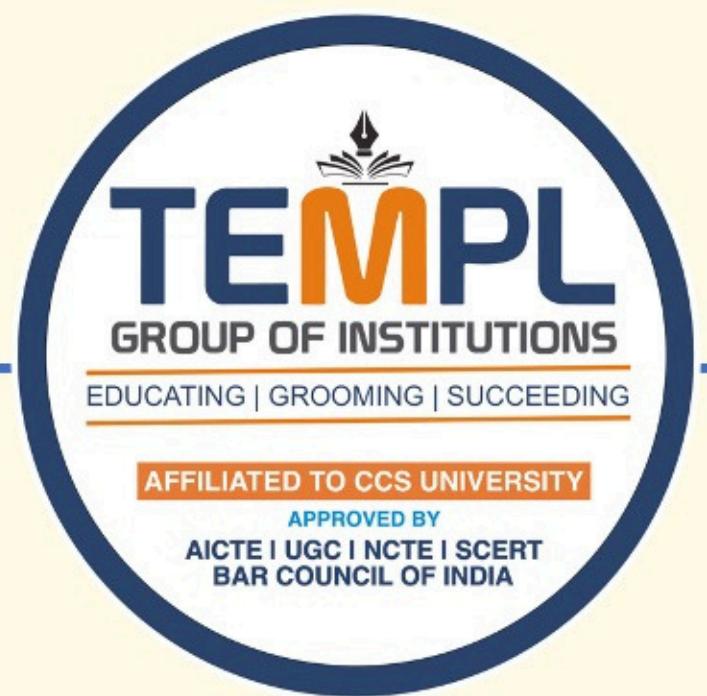
DIFFERENTIAL EQUATION



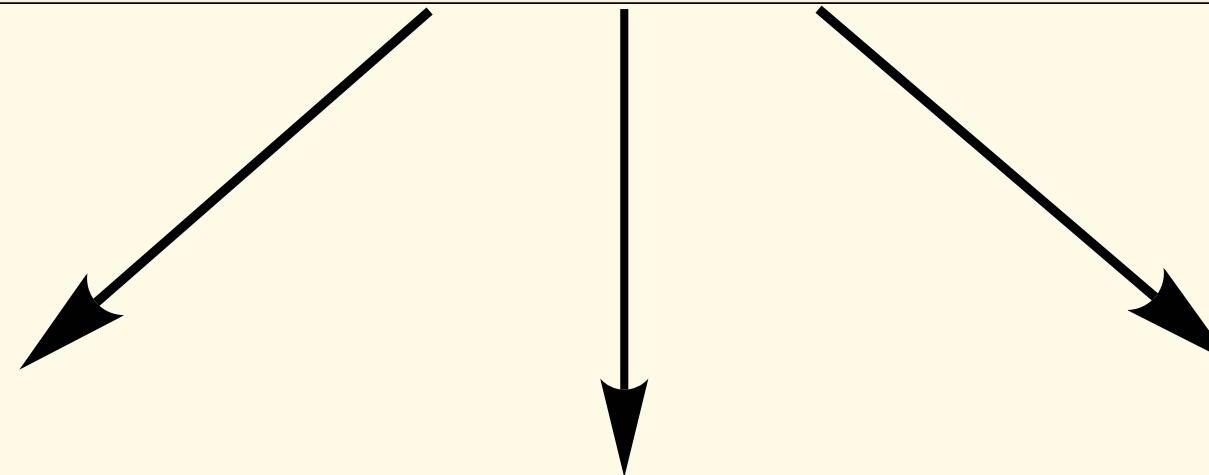
A differential equation is an equation that involves a function and its derivative(s).

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

SOLUTION OF DIFFERENTIAL EQUATION



SOLUTION OF DIFFERENTIAL EQUATION



Euler's Method

Picard's Method

Runge Kutta Method

EULER'S METHOD

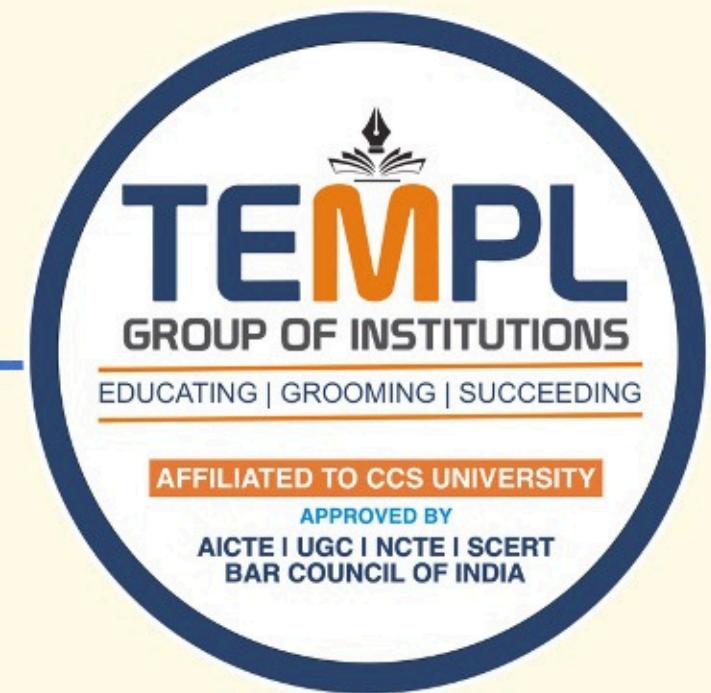


Euler's method is the simplest numerical method to solve a first order differential equation.

It uses the concept of slope to predict the next value step by step.

The Euler Method was founded by the Swiss mathematician Leonhard Euler around the 18th century.

FORMULA OF EULER'S METHOD



For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

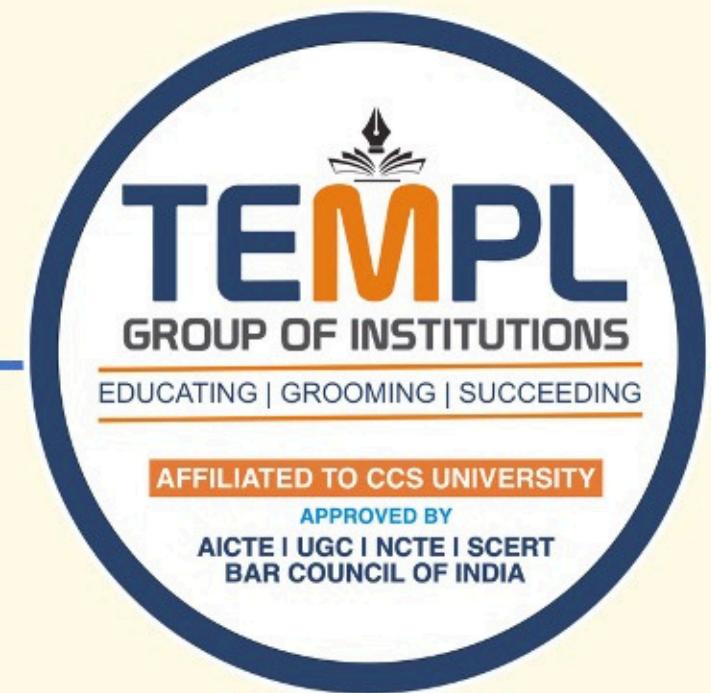
Euler's formula is

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

where,

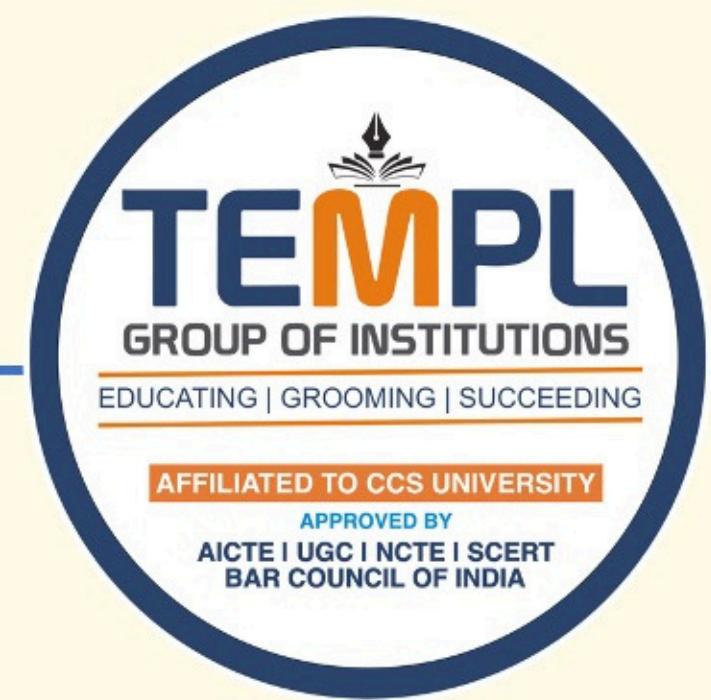
- x_0, y_0 = initial values
- h = step size
- $f(x, y)$ = given function (slope)

STEPS OF EULER'S METHOD



- » **Start with initial condition (x_0, y_0)**
- » **Choose a step size h .**
- » **Compute slope, $f(x_n, y_n)$.**
- » **Use formula,**
$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$
- » **Repeat for required steps.**

EXAMPLE OF EULER'S METHOD



The initial value problem

$$\frac{dy}{dx} = x - y, \quad y(0) = 1$$

with step size $h = 0.1$.

Find approximations up to $x = 0.3$.

Then

$$y_{n+1} = y_n + h f(x_n, y_n)$$

where $f(x, y) = x - y$.

$$x_0 = 0.0, \quad y_0 = 1.000.$$

Compute step by step

n	x_n	y_n	$f(x_n, y_n) = x_n - y_n$	$y_{n+1} = y_n + h f$
0	0.0	1.000	$0.0 - 1.000 = -1.000$	$1.000 + 0.1(-1.000) = 0.900$
1	0.1	0.900	$0.1 - 0.900 = -0.800$	$0.900 + 0.1(-0.800) = 0.820$
2	0.2	0.820	$0.2 - 0.820 = -0.620$	$0.820 + 0.1(-0.620) = 0.758$

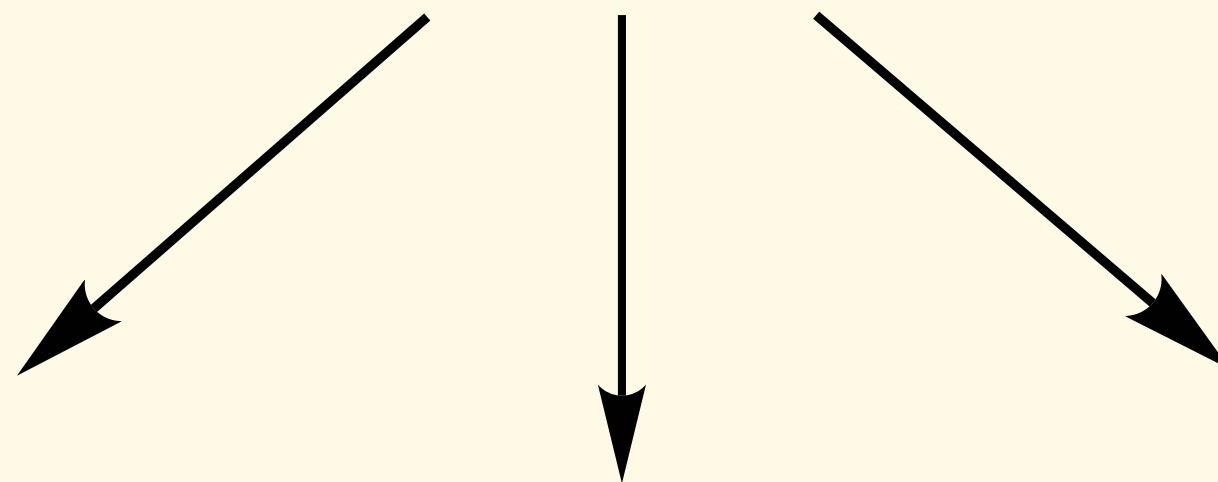
So the Euler approximations are

$$y(0.1) \approx 0.900, \quad y(0.2) \approx 0.820, \quad y(0.3) \approx 0.758$$

TYPES OF EULER'S METHOD



TYPES OF EULER'S METHOD



**Simple Euler's Method
(Basic Method)**

**Improved Euler's Method
(Heun's Method)**

**Modified Euler's Method
(Midpoint Method)**

SIMPLE (OR BASIC) EULER'S METHOD



Euler's method is the simplest numerical method to solve a first order differential equation of the form

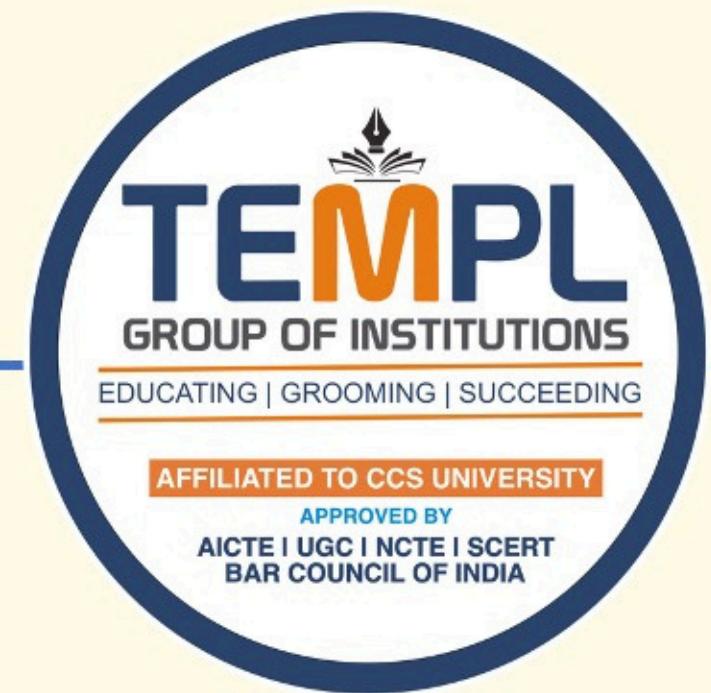
$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

It finds the approximate value of y at successive points.

$$y_{n+1} = y_n + h f(x_n, y_n)$$

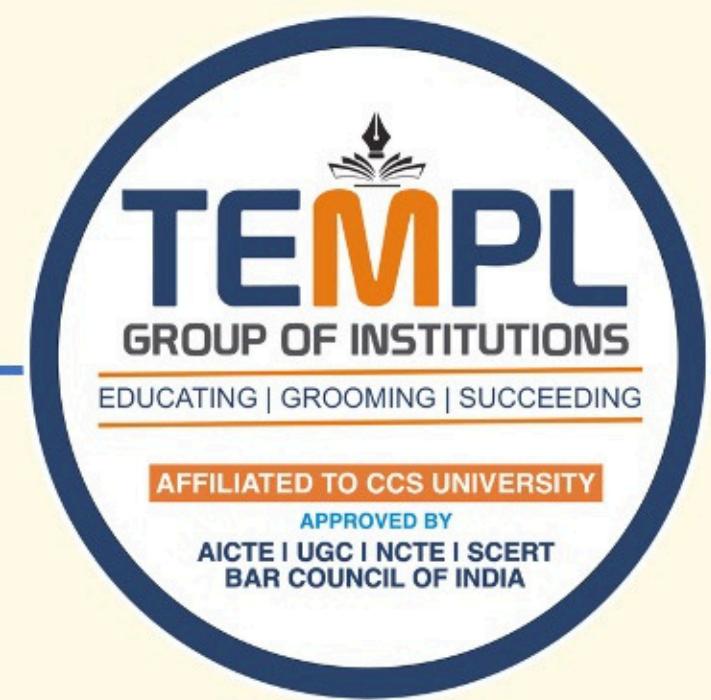
Easy but less accurate (error is proportional to step size h).

STEPS OF EULER'S METHOD



- » **Start with initial condition (x_0, y_0)**
- » **Choose a step size h .**
- » **Compute slope, $f(x_n, y_n)$.**
- » **Use formula,**
$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$
- » **Repeat for required steps.**

EXAMPLE OF EULER'S METHOD



The initial value problem

$$\frac{dy}{dx} = x - y, \quad y(0) = 1$$

with step size $h = 0.1$.

Find approximations up to $x = 0.3$.

Then

$$y_{n+1} = y_n + h f(x_n, y_n)$$

where $f(x, y) = x - y$.

$$x_0 = 0.0, \quad y_0 = 1.000.$$

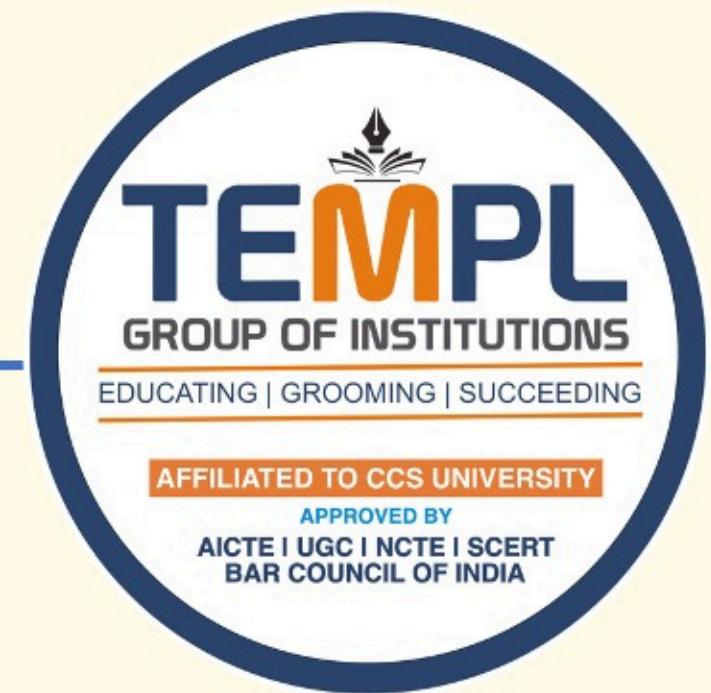
Compute step by step

n	x_n	y_n	$f(x_n, y_n) = x_n - y_n$	$y_{n+1} = y_n + h f$
0	0.0	1.000	$0.0 - 1.000 = -1.000$	$1.000 + 0.1(-1.000) = 0.900$
1	0.1	0.900	$0.1 - 0.900 = -0.800$	$0.900 + 0.1(-0.800) = 0.820$
2	0.2	0.820	$0.2 - 0.820 = -0.620$	$0.820 + 0.1(-0.620) = 0.758$

So the Euler approximations are

$$y(0.1) \approx 0.900, \quad y(0.2) \approx 0.820, \quad y(0.3) \approx 0.758$$

IMPROVED EULER'S METHOD (HEUN'S METHOD)



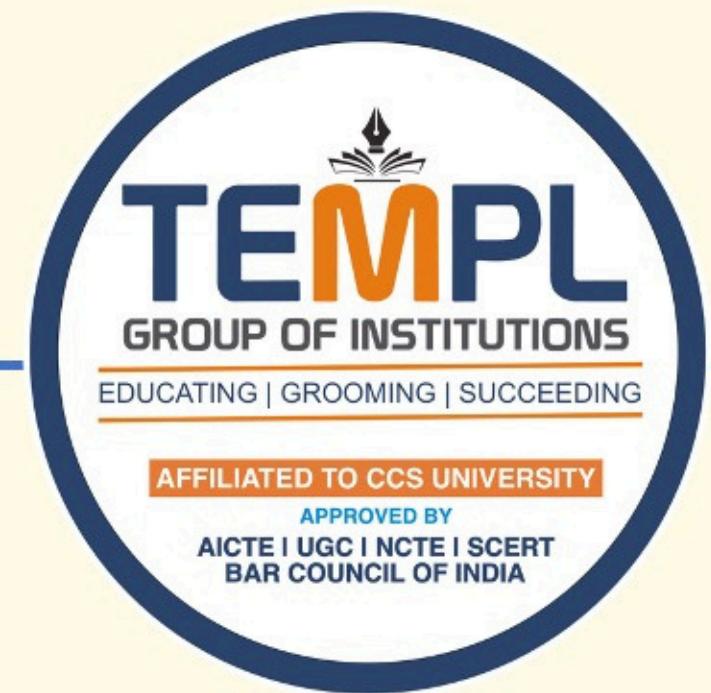
Improved Euler's Method is a correction of the simple Euler's method.

It improves accuracy by taking the average slope at the beginning and end of the interval.

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))]$$

More accurate than Euler because it uses two slope values.

STEPS OF IMPROVED EULER'S METHOD



- Start with initial condition (x_0, y_0) .
- Choose a step size h .
- Predictor Step (Euler's formula)
- Corrector Step (Average slope)
- Repeat for required steps.

$$y^* = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y^*)]$$

EXAMPLE OF IMPROVED EULER'S METHOD



The initial value problem

$$\frac{dy}{dx} = x - y, \quad y(0) = 1$$

with step size $h = 0.1$.

Find approximations up to $x = 0.3$.

Then

Predictor

$$y^* = y_n + h f(x_n, y_n)$$

Corrector

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y^*)]$$

Compute step by step

n	x_n	y_n	$f(x_n, y_n) = x_n - y_n$	Predictor $y^* \text{ (calc)}$	$f(x_{n+1}, y^*)$	Corrector $y_{n+1} \text{ (calc)}$
0	0.0	1.00000	-1.00000	$0.900000 (1 + 0.1 \cdot (-1)) = 0.900000$	-0.800000	$0.910000 (1 + \frac{0.1}{2}(-1 + -0.8)) = 0.910000$
1	0.1	0.91000	-0.81000	$0.829000 (0.91 + 0.1 \cdot (-0.81)) = 0.829000$	-0.629000	$0.838050 (0.91 + \frac{0.1}{2}(-0.81 + -0.629)) = 0.838050$
2	0.2	0.838050	-0.638050	$0.774245 (0.83805 + 0.1 \cdot (-0.63805)) = 0.774245$	-0.474245	$0.782435 (0.83805 + \frac{0.1}{2}(-0.63805 + -0.474245)) = 0.78243525 \rightarrow 0.782435$

So the Euler approximations are

$$y(0.1) \approx 0.91000, \quad y(0.2) \approx 0.83805, \quad y(0.3) \approx 0.78243$$

MODIFIED EULER'S METHOD (MIDPOINT METHOD)



Modified Euler's Method is another correction form of Euler's method.

It calculates the slope at the midpoint of the interval for better accuracy.

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

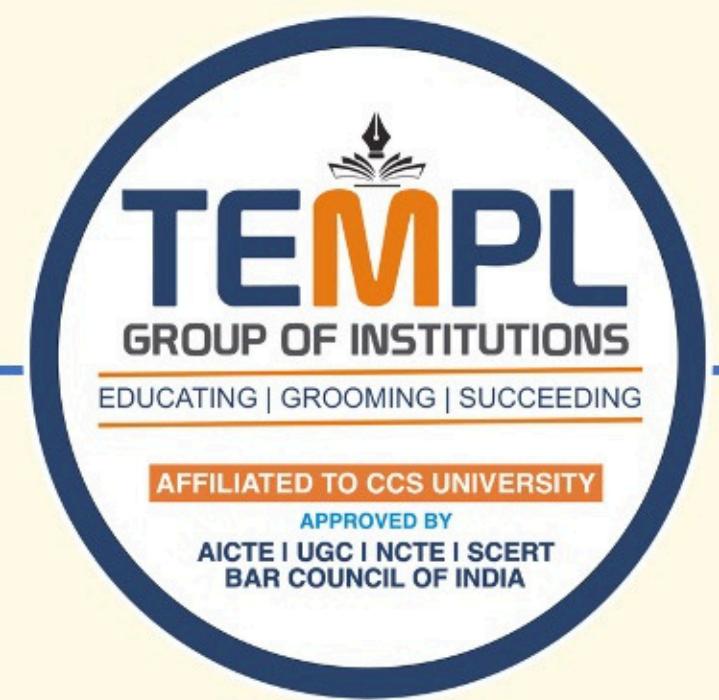
Accuracy better than simple Euler but slightly different from Heun's method.

STEPS OF MODIFIED EULER'S METHOD



- » Start with initial condition (x_0, y_0) .
- » Choose a step size h .
- » Compute slope at the beginning (Predictor).
$$k_1 = f(x_n, y_n)$$
- » Compute slope at midpoint (Corrector).
$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$
- » Use formula to update.
$$y_{n+1} = y_n + h \cdot k_2$$
- » Repeat for required steps.

EXAMPLE OF MODIFIED EULER'S METHOD



The initial value problem

$$\frac{dy}{dx} = x - y, \quad y(0) = 1$$

with step size $h = 0.1$.

Find approximations up to $x = 0.3$.

Then

Predictor

$$k_1 = f(x_n, y_n)$$

Corrector

$$y^* = y_n + \frac{h}{2} k_1$$

$$k_2 = f\left(x_n + \frac{h}{2}, y^*\right)$$

So

$$y_{n+1} = y_n + h k_2$$

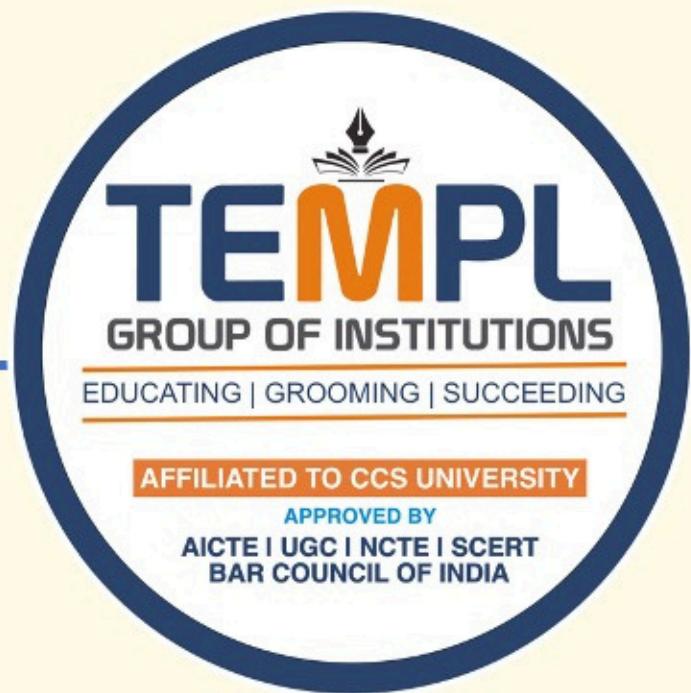
Compute step by step

n	x_n	y_n	$k_1 = f(x_n, y_n) = x_n - y_n$	$y^* = y_n + \frac{h}{2} k_1$	$k_2 = f(x_n + \frac{h}{2}, y^*)$	$y_{n+1} = y_n + h k_2$
0	0.0	1.000000	-1.000000	0.950000	-0.900000	0.910000
1	0.1	0.910000	-0.810000	0.869500	-0.719500	0.838050
2	0.2	0.838050	-0.638050	0.8061475	-0.5561475	0.78243525

So the Euler approximations are

$$y(0.1) \approx 0.9100, \quad y(0.2) \approx 0.8380, \quad y(0.3) \approx 0.7824$$

COMPARISON OF EULER, IMPROVED EULER & MODIFIED EULER METHODS



COMPARISON OF EULER, IMPROVED EULER & MODIFIED EULER METHODS

EULER'S METHOD

$$y_{n+1} = y_n + h f(x_n, y_n)$$

Initial slope only

Accuracy Higher

IMPROVED EULER

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_n + h f(x_n, y_n))]$$

Start & End slope avg.

Accuracy Higher

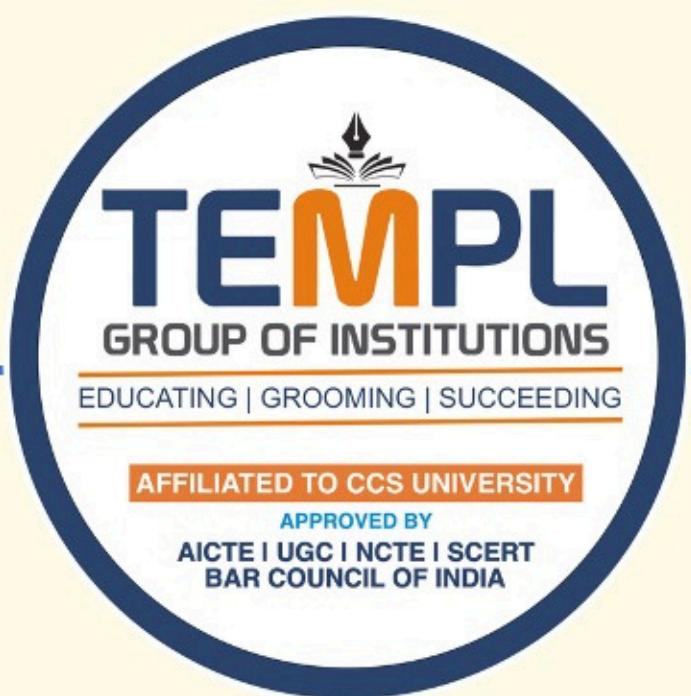
MODIFIED EULER

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

Midpoint slope

Accuracy Low

PICARD'S METHOD

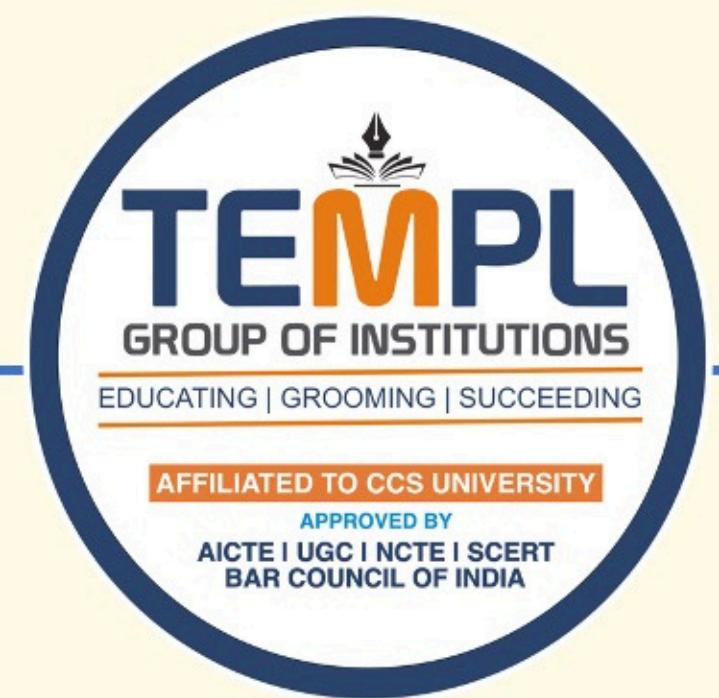


Picard's Method is an iterative method used to obtain an approximate solution of a first order ordinary differential equation.

The method is based on successive approximations.

The Picard's Method was founded by the French mathematician Charles Émile Picard around the 19th century.

FORMULA OF PICARD'S METHOD



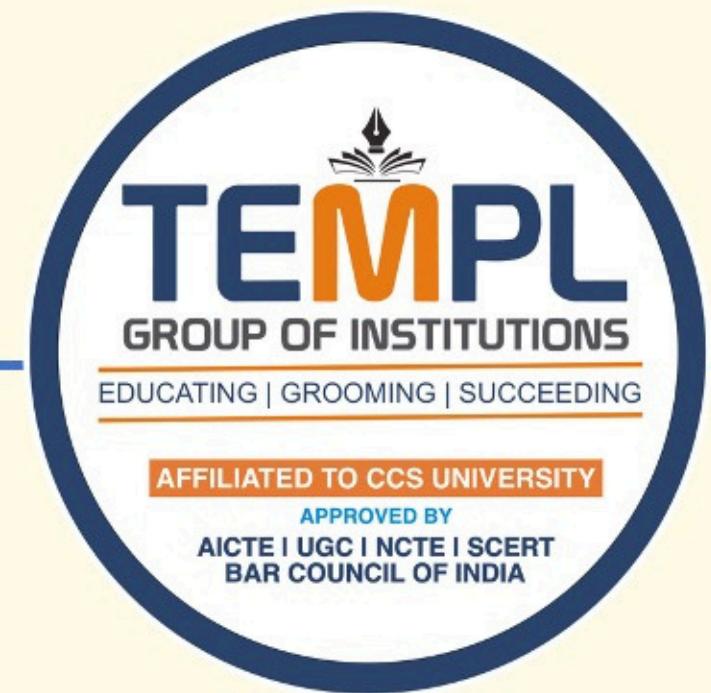
For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Picard's formula is

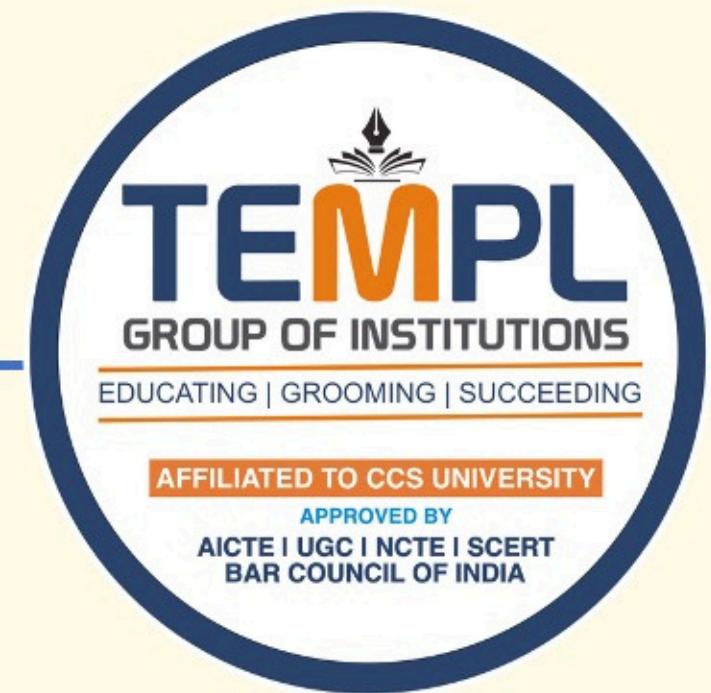
$$y_{n+1}(x) = y_0 + \int_{x_0}^x f(t, y_n(t)) dt$$

STEPS OF PICARD'S METHOD



- » Write the differential equation in integral form.
- » Choose an initial approximation $y_0(x)$.
- » Substitute $y_0(x)$ in the integral to get $y_1(x)$.
- » Repeat the process to obtain $y_2(x)$, $y_3(x)$,...
- » Continue until the required accuracy is achieved.

RUNGE-KUTTA (RK) METHOD

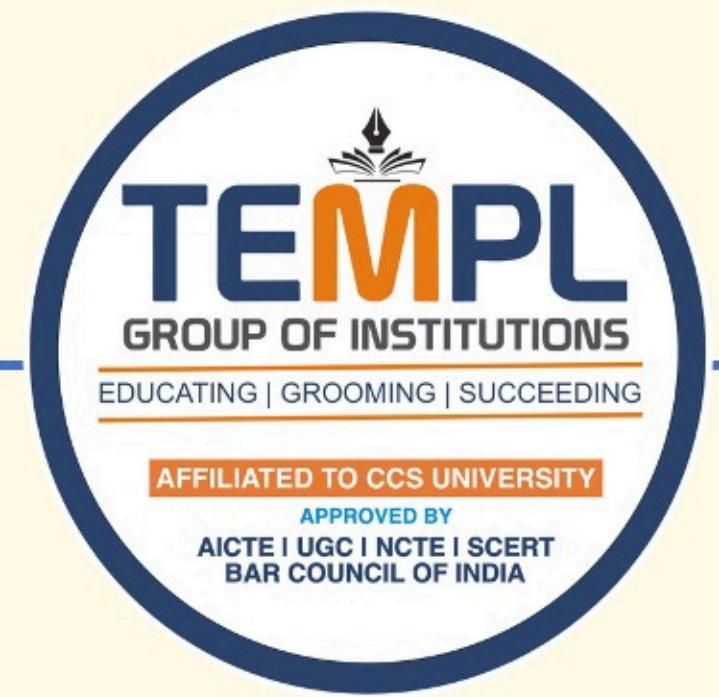


The Runge-Kutta (RK) method is a numerical technique used to obtain an approximate solution of first order ordinary differential equations.

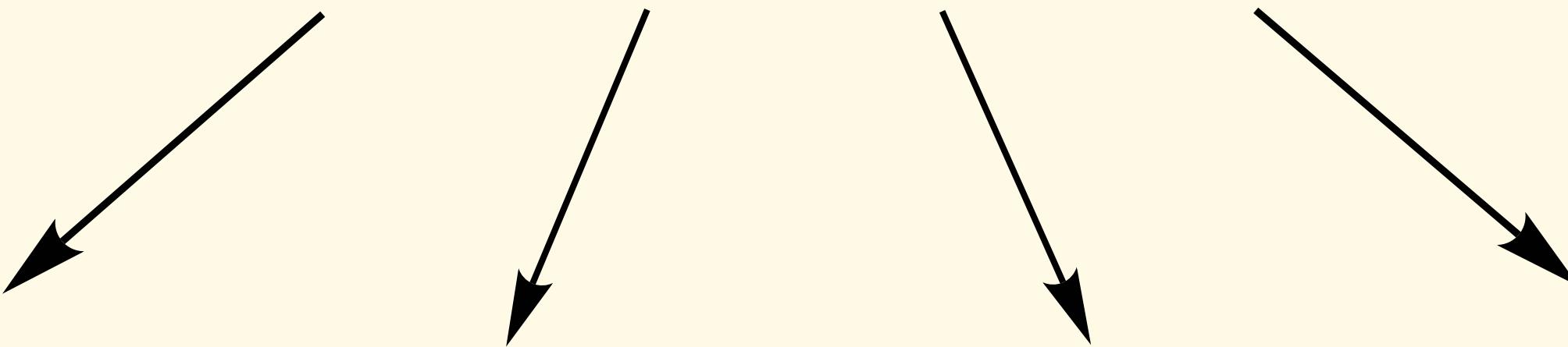
The method improves accuracy by taking a weighted average of slopes at different points within each step.

The Runge-Kutta (RK) method was founded by the German mathematicians Carl Runge and Martin Kutta around the 20th century .

TYPES OF RUNGE-KUTTA (RK) METHOD



TYPES OF RUNGE-KUTTA (RK) METHOD



**First order
R-K method**

**Second order
R-K method**

**Third order
R-K method**

**Fourth order
R-K method**

FORMULA OF FIRST ORDER R-K METHOD



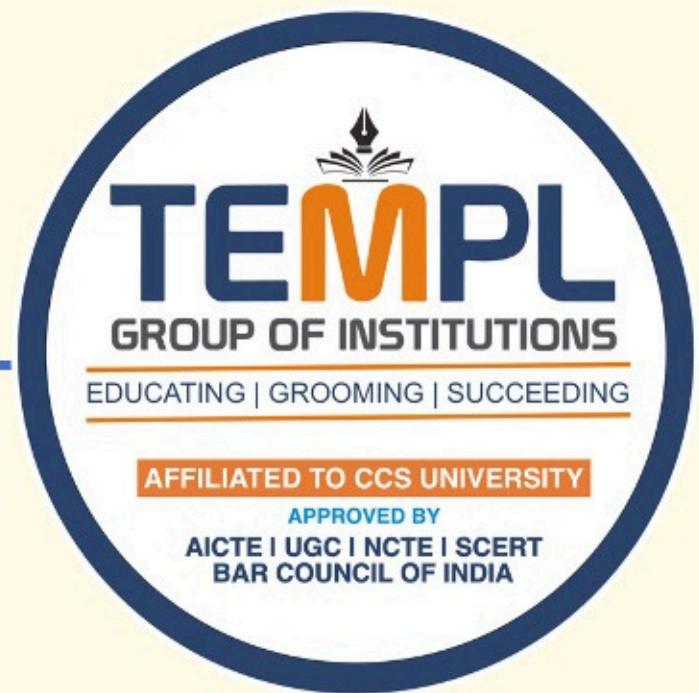
For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

First Order Runge-Kutta's formula is

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

FORMULA OF SECOND ORDER R-K METHOD



For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

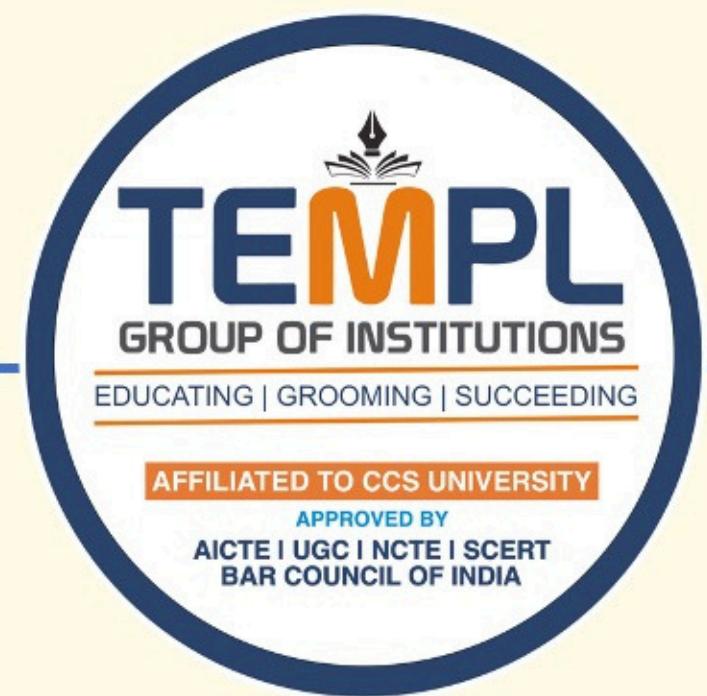
Second Order Runge-Kutta's formula is

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$y_{n+1} = y_n + k_2$$

FORMULA OF THIRD ORDER R-K METHOD



For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Third Order Runge-Kutta's formula is

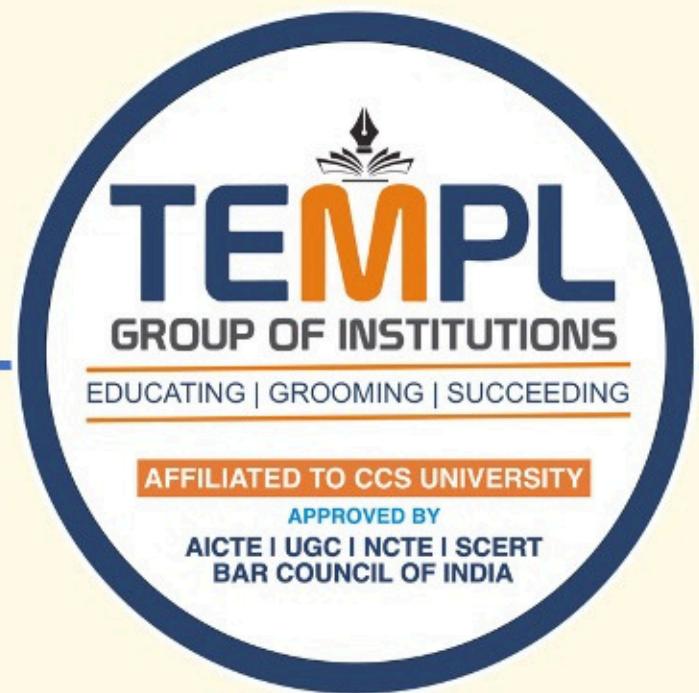
$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h f(x_n + h, y_n - k_1 + 2k_2)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

FORMULA OF FOURTH ORDER R-K METHOD



For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Fourth Order Runge-Kutta's formula is

$$k_1 = h f(x_n, y_n)$$

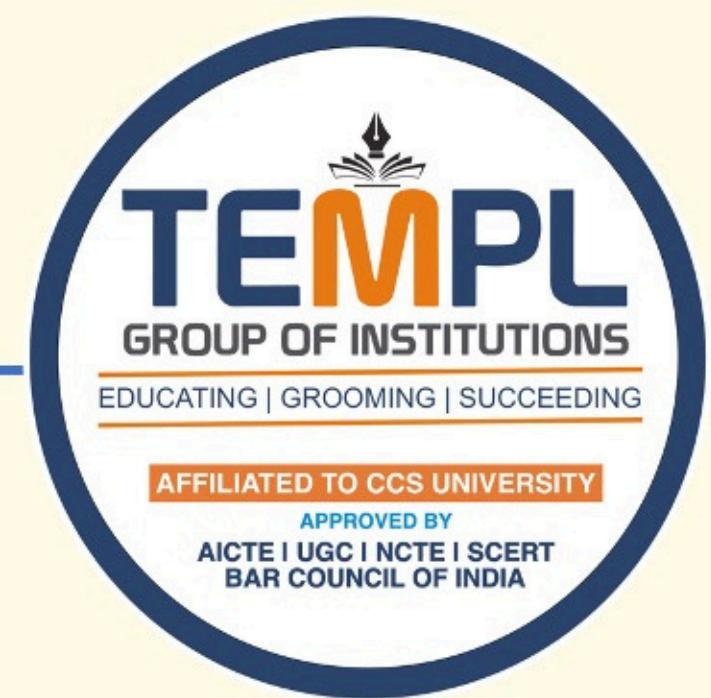
$$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

RUNGE-KUTTA (RK) METHOD



QUICK EXAM ORIENTED SUMMARY TABLE

RK Method	Order	Accuracy	Syllabus Importance
RK-1	1st	Low	Basic
RK-2	2nd	Moderate	Important
RK-3	3rd	Good	Sometimes asked
RK-4	4th	Very High	Most Important