NUMERICAL METHODS

VAISHALI
[Faculty of Mathematics]

UNIT-5

SOLUTION OF DIFFERENTIAL EQUATION

DIFFERENTIAL EQUATION

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A differential equation is an equation that involves a function and its derivative(s).

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

SOLUTION OF DIFFERENTIAL EQUATION

SOLUTION OF DIFFERENTIAL EQUATION

- Euler's Method
- Picard's Mrthod
- Runge Kutta Method

EULER'S METHOD

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Euler's method is the simplest numerical method to solve a first order differential equation.

It uses the concept of slope to predict the next value step by step.

The Euler Method was founded by the Swiss mathematician Leonhard Euler around the 18th century.

FORMULA OF EULER'S METHOD

For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Euler's formula is

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

where,

- **x0,y0 = initial values**
- **h = step size**
- **f(x,y) = given function (slope)**

STEPS OF EULER'S METHOD

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- Start with initial condition (x0,y0)
- Choose a step size h.
- Compute slope, f(xn,yn).
- Use formula, $y_{n+1} = y_n + h \cdot f(x_n, y_n)$
- Repeat for required steps.

EXAMPLE OF EULER'S METHOD

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The initial value problem

$$\frac{dy}{dx} = x - y, \qquad y(0) = 1$$

with step size $h = 0.1$.

Find approximations up to $x = 0.3$.

Then

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

where $f(x, y) = x - y$.

$$x_0 = 0.0,\ y_0 = 1.000.$$

Compute step by step

n	x_n	y_n	$f(x_n, y_n) = x_n - y_n$	$y_{n+1} = y_n + hf$
0	0.0	1.000	$0.0 - 1.000 = -1.000$	$1.000 + 0.1(-1.000) = 0.900$
1	0.1	0.900	$0.1 - 0.900 = -0.800$	$0.900 + 0.1(-0.800) = 0.820$
2	0.2	0.820	$0.2 - 0.820 = -0.620$	$0.820 + 0.1(-0.620) = 0.758$

So the Euler approximations are

$y(0.1) \approx 0.900, \qquad y(0.2) \approx 0.820, \qquad y(0.3) \approx 0.758$

TYPES OF EULER'S METHOD

TYPES OF EULER'S METHOD

- Simple Euler's Method (Basic Method)
- Improved Euler's Method (Heun's Method)
- Modified Euler's Method (Midpoint Method)

SIMPLE (OR BASIC) EULER'S METHOD

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Euler's method is the simplest numerical method to solve a first order differential equation of the form

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

It finds the approximate value of y at successive points.

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Easy but less accurate (error is proportional to step size h).

STEPS OF EULER'S METHOD

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- Start with initial condition (x0,y0)
- Choose a step size h.
- Compute slope, f(xn,yn).
- Use formula, $$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$
- Repeat for required steps.

EXAMPLE OF EULER'S METHOD

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The initial value problem

$$\frac{dy}{dx} = x - y, \qquad y(0) = 1$$

with step size $h = 0.1$.

Find approximations up to $x = 0.3$.

Then

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

where $f(x, y) = x - y$.

$x_0 = 0.0,\ y_0 = 1.000$.

Compute step by step

n	x_n	y_n	$f(x_n, y_n) = x_n - y_n$	$y_{n+1} = y_n + hf$
0	0.0	1.000	$0.0 - 1.000 = -1.000$	$1.000 + 0.1(-1.000) = 0.900$
1	0.1	0.900	$0.1 - 0.900 = -0.800$	$0.900 + 0.1(-0.800) = 0.820$
2	0.2	0.820	$0.2 - 0.820 = -0.620$	$0.820 + 0.1(-0.620) = 0.758$

So the Euler approximations are

$y(0.1) \approx 0.900, \quad y(0.2) \approx 0.820, \quad y(0.3) \approx 0.758$

IMPROVED EULER'S METHOD (HEUN'S METHOD)

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Improved Euler's Method is a correction of the simple Euler's method.

It improves accuracy by taking the average slope at the beginning and end of the interval.

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\right]$$

More accurate than Euler because it uses two slope values.

STEPS OF IMPROVED EULER'S METHOD

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- Start with initial condition (x0,y0).
- Choose a step size h.
- Predictor Step (Euler's formula) $$y^* = y_n + hf(x_n, y_n)$$
- Corrector Step (Average slope) $$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*)\right]$$
- Repeat for required steps.

EXAMPLE OF IMPROVED EULER'S METHOD

The initial value problem

$$\frac{dy}{dx} = x - y, \qquad y(0) = 1$$

with step size $h = 0.1.$

Find approximations up to $x = 0.3.$

Then

Predictor $y^* = y_n + hf(x_n, y_n)$

Corrector $y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*)\right]$

Compute step by step

n	x_n	y_n	$f(x_n, y_n) = x_n - y_n$	Predictor y^* (calc)	$f(x_{n+1}, y^*)$	Corrector y_{n+1} (calc)
0	0.0	1.000000	-1.000000	0.900000 ($1 + 0.1 \cdot (-1) = 0.900000$)	-0.800000	0.910000 ($1 + \frac{0.1}{2}(-1 + -0.8) = 0.910000$)
1	0.1	0.910000	-0.810000	0.829000 ($0.91 + 0.1 \cdot (-0.81) = 0.829000$)	-0.629000	0.838050 ($0.91 + \frac{0.1}{2}(-0.81 + -0.629) = 0.838050$)
2	0.2	0.838050	-0.638050	0.774245 ($0.83805 + 0.1 \cdot (-0.63805) = 0.774245$)	-0.474245	0.782435 ($0.83805 + \frac{0.1}{2}(-0.63805 + -0.474245) = 0.78243525$ → 0.782435)

So the Euler approximations are

$y(0.1) \approx 0.91000, \quad y(0.2) \approx 0.83805, \quad y(0.3) \approx 0.78243$

MODIFIED EULER'S METHOD (MIDPOINT METHOD)

Modified Euler's Method is another correction form of Euler's method.

It calculates the slope at the midpoint of the interval for better accuracy.

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

Accuracy better than simple Euler but slightly different from Heun's method.

STEPS OF MODIFIED EULER'S METHOD

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- Start with initial condition (x0,y0).
- Choose a step size h.
- Compute slope at the beginning (Predictor). $k_1 = f(x_n, y_n)$
- Compute slope at midpoint (Corrector). $k_2 = f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2}k_1\right)$
- Use formula to update. $y_{n+1} = y_n + h \cdot k_2$
- Repeat for required steps.

EXAMPLE OF MODIFIED EULER'S METHOD

The initial value problem

$$\frac{dy}{dx} = x - y, \qquad y(0) = 1$$

with step size $h = 0.1.$

Find approximations up to $x = 0.3.$

Then

Predictor $k_1 = f(x_n, y_n)$

Corrector $y^* = y_n + \frac{h}{2} k_1$ $\quad k_2 = f\left(x_n + \frac{h}{2},\ y^*\right)$

So $y_{n+1} = y_n + h\,k_2$

Compute step by step

n	x_n	y_n	$k_1 = f(x_n, y_n) = x_n - y_n$	$y^* = y_n + \frac{h}{2}k_1$	$k_2 = f(x_n + \frac{h}{2}, y^*)$	$y_{n+1} = y_n + hk_2$
0	0.0	1.000000	-1.000000	0.950000	-0.900000	0.910000
1	0.1	0.910000	-0.810000	0.869500	-0.719500	0.838050
2	0.2	0.838050	-0.638050	0.8061475	-0.5561475	0.78243525

So the Euler approximations are

$y(0.1) \approx 0.9100, \qquad y(0.2) \approx 0.8380, \qquad y(0.3) \approx 0.7824$

COMPARISON OF EULER, IMPROVED EULER & MODIFIED EULER METHODS

COMPARISON OF EULER, IMPROVED EULER & MODIFIED EULER METHODS

EULER'S METHOD	IMPROVED EULER	MODIFIED EULER
$y_{n+1} = y_n + hf(x_n, y_n)$	$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))]$	$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)\right)$
Initial slope only	Start & End slope avg.	Midpoint slope
Accuracy Higher	Accuracy Higher	Accuracy Low

PICARD'S METHOD

Picard's Method is an iterative method used to obtain an approximate solution of a first order ordinary differential equation.

The method is based on successive approximations.

The Picard's Method was founded by the French mathematician Charles Émile Picard around the 19th century.

FORMULA OF PICARD'S METHOD

For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Picard's formula is

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt$$

STEPS OF PICARD'S METHOD

- Write the differential equation in integral form.
- Choose an initial approximation $y_0(x)$.
- Substitute $y_0(x)$ in the integral to get $y_1(x)$.
- Repeat the process to obtain $y_2(x)$, $y_3(x)$,…
- Continue until the required accuracy is achieved.

RUNGE-KUTTA (RK) METHOD

The Runge–Kutta (RK) method is a numerical technique used to obtain an approximate solution of first order ordinary differential equations.

The method improves accuracy by taking a weighted average of slopes at different points within each step.

> The Runge–Kutta (RK) method was founded by the German mathematicians Carl Runge and Martin Kutta around the 20^{th} century .

TYPES OF RUNGE-KUTTA (RK) METHOD

TYPES OF RUNGE–KUTTA (RK) METHOD

- First order R–K method
- Second order R–K method
- Third order R–K method
- Fourth order R–K method

FORMULA OF FIRST ORDER R-K METHOD

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For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

First Order Runge–Kutta's formula is

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

FORMULA OF SECOND ORDER R-K METHOD

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For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Second Order Runge–Kutta's formula is

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$y_{n+1} = y_n + k_2$$

FORMULA OF THIRD ORDER R-K METHOD

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For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Third Order Runge-Kutta's formula is

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_n + h, y_n - k_1 + 2k_2)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

FORMULA OF FOURTH ORDER R-K METHOD

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For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Fourth Order Runge–Kutta's formula is

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

RUNGE-KUTTA (RK) METHOD

QUICK EXAM ORIENTED SUMMARY TABLE

RK Method	Order	Accuracy	Syllabus Importance
RK–1	1st	Low	Basic
RK–2	2nd	Moderate	Important
RK–3	3rd	Good	Sometimes asked
RK–4	4th	Very High	**Most Important**